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A regularity result for polyharmonic maps with higher integrability

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Abstract We prove a regularity result for critical points of the polyharmonic energy $E(u) = \int_\Omega |\nabla^k u|^2 dx$ in $W^{k,2p}(\Omega, \mathcal{N})$ with $k \in \mathbb{N}$ and $p > 1$. Our proof is based on a Gagliardo–Nirenberg-type estimate and avoids the moving frame technique. In view of the monotonicity formulae for stationary harmonic and biharmonic maps, we infer partial regularity in theses cases.

Keywords Polyharmonic maps · Harmonic maps · Biharmonic maps · Gagliardo–Nirenberg inequality

1 Introduction

Let $\Omega \subset \mathbb{R}^m$ be a smooth open bounded domain and $\mathcal{N}$ a smooth closed Riemannian manifold isometrically embedded in $\mathbb{R}^N$. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, we define the Sobolev spaces

$$W^{k,p}(\Omega, \mathcal{N}) := \{u \in W^{k,p}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega\}$$

equipped with the topology inherited from the topology of the linear Sobolev space $W^{k,p}(\Omega, \mathbb{R}^N)$.

For $u \in W^{1,2}(\Omega, \mathcal{N})$, we consider the Dirichlet energy

$$D(u) := \int_\Omega |\nabla u|^2 dx.$$

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Critical points of $D(\cdot)$ in $W^{1,2}(\Omega, \mathcal{N})$ with respect to compactly supported variations in the target manifold are called weakly harmonic. If $u \in W^{1,2}(\Omega, \mathcal{N})$ is weakly harmonic, and in addition, is a critical point with respect to compactly supported variations in the domain manifold, then u is called stationary harmonic.

Regularity properties of weakly harmonic maps have been intensely studied during the last decades. For $m = 2$, Morrey [16] showed in 1948 that every minimizing map $u \in W^{1,2}(\Omega, \mathcal{N})$ belongs to $C^\infty(\Omega, \mathcal{N})$. The regularity problem for general critical points of the harmonic energy functional had remained open for a long time. In 1981, again for the case $m = 2$, Grüter [11] proved smoothness of conformal weakly harmonic maps. Schoen [24] introduced the notion of stationary harmonic maps and extended Grüter's result to this class. Finally, Hélein [12] showed that every weakly harmonic map in the case $m = 2$ is smooth. For $m \geq 3$, more complex phenomena show up. Schoen and Uhlenbeck [25] showed that if $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$ is energy minimizing, then u is smooth except on a closed subset S with Hausdorff dimension $dim_{\mathcal{H}}(S) \leq m-3$. In particular, for $m = 3$, they show that S is reduced to at most isolated points. This result is optimal since the radial projection from B^m into S^{m-1} is a minimizing map for $m \geq 3$, as shown by Brezis et al. [4] for $m = 3$ and Lin [14] for $m \geq 3$. On the other hand, Rivière [21] proved the existence of everywhere discontinuous weakly harmonic maps. For the intermediate class of stationary harmonic maps, Evans [6] showed partial regularity for maps into the sphere and Bethuel [3] generalized this result for arbitrary target manifolds. Their proofs rely on a monotonicity formula for stationary harmonic maps adapted from Price [19].

Similar questions were studied for weakly (extrinsic) biharmonic and stationary biharmonic maps, which are the critical points of the Hessian energy functional

$$B(u) = \int_\Omega |\Delta u|^2 dx$$

in $W^{2,2}(\Omega, \mathcal{N})$. Chang et al. [5] showed smoothness for weakly biharmonic maps into the sphere and $m \leq 4$ (see also Strzelecki [28]), and asserted partial regularity for stationary biharmonic maps into the sphere and $m \geq 5$. Wang generalized these results for arbitrary target manifolds in [31] and [32]. Once again, a monotonicity formula derived from the stationarity assumption is crucial in the proof of partial regularity for $m \geq 5$. This monotonicity formula appeared in [5] for sufficiently regular maps. However, a rigorous proof in the case of stationary biharmonic maps of class $W^{2,2}(\Omega, \mathcal{N})$, concluding the partial regularity, results in the above mentioned papers, first appeared in Angelsberg [2].

In the case of target manifolds without symmetry, another important tool for proving (partial) regularity for harmonic and biharmonic maps is the technique of moving frames. This was introduced for harmonic maps in two dimensions by Hélein [12], applied to stationary harmonic maps by Bethuel [3] and later to (stationary) biharmonic maps by Wang in [31] and [32]. Only very recently, Rivière [22] succeeded in rephrasing the harmonic map system as a conservation law when $m = 2$, allowing him (amongst other results) to give a direct proof of regularity of weakly harmonic maps in two dimensions avoiding the use of moving frames. After that, Rivière and Struwe [23] developed a related gauge-theoretic approach to prove partial regularity in higher dimensions. Moreover, this new approach allowed the authors to reduce Hélein's $\mathcal{C}^5$-assumption on the target manifold to $\mathcal{C}^2$, which seems to be the natural assumption to ensure that the second fundamental form is well defined. Finally, Lamm and Rivière [13] showed smoothness for weakly biharmonic maps in four dimensions avoiding moving frames and Struwe [27] proved partial regularity for stationary biharmonic maps in higher dimensions via gauge theory.

Strengthening the natural hypotheses for the regularity of a stationary biharmonic map u slightly by assuming some higher integrability of the leading order derivative, we here show similar partial regularity results for biharmonic maps without using moving frames. Our method is not restricted to this fourth order problem and also provides regularity results for polyharmonic maps, which are defined below.

For $k \in \mathbb{N}$ and $u \in W^{k,2}(\Omega, \mathcal{N})$, we consider the k-harmonic energy functional

$$E(u) = \int\limits_{\Omega} |\nabla^k u|^2 dx.$$

Define the *BMO* space and the Morrey spaces $M^{p,\lambda}$ for $1 \leq p < \infty$, $0 < \lambda \leq m$ as

$$BMO(\Omega) := \left\{ u \in L^1(\Omega) : [u]_{BMO(\Omega)} := \sup_{B_r \subset \mathbb{R}^m} \left\{ r^{-m} \int\limits_{B_r \cap \Omega} |u - \overline{u}_{B_r \cap \Omega}| dx \right\} < \infty \right\}$$

and

$$M^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) : [u]^p_{M^{p,\lambda}(\Omega)} := \sup_{B_r \subset \mathbb{R}^m} \left\{ r^{\lambda - m} \int\limits_{B_r \cap \Omega} |u|^p dx \right\} < \infty \right\},$$

where $\overline{u}_{B_r} := \fint_{B_r} u dx$ denotes the average of u on B_r.

Definition 1 A map $u \in W^{k,2}(\Omega, \mathcal{N})$ is called weakly k-harmonic if u is a critical point of the k-harmonic energy functional with respect to compactly supported variations on $\mathcal{N}$. That is, if for all $\xi \in C_0^\infty(\Omega, \mathbb{R}^N)$, we have

$$\left.\frac{d}{dt}\right|_{t=0} E(\pi(u + t\xi)) = 0,$$

where π denotes the nearest point projection onto $\mathcal{N}$.

Definition 2 A weakly k-harmonic map in $W^{k,2}(\Omega, \mathcal{N})$ is called stationary k-harmonic if, in addition, u is a critical point of the k-harmonic energy $E(\cdot)$ with respect to compactly supported variations on the domain manifold, i.e., if

$$\left.\frac{d}{dt}\right|_{t=0} E\left(u \circ (id + t\xi)\right) = 0 \quad \text{for all } \xi \in C_0^\infty(\Omega, \mathbb{R}^m), \tag{1}$$

where id denotes the identity map.

Remark 1.1 (Stationary) 1-harmonic maps are the (stationary) harmonic maps. Observing that $|\nabla^2 u|^2$ and $|\Delta u|^2$ only differ by a divergence term, we conclude that the (stationary) 2-harmonic maps are precisely the (stationary) biharmonic maps.

Our main result then reads

Theorem 1.1 *For $p > 1$ and $2kp \leq m$, let $u \in W^{k,2p}(\Omega, \mathcal{N})$ be weakly k-harmonic. There exists $\epsilon > 0$, such that for each point $x_0 \in \Omega$ for which there exists some $r_0 > 0$ with*

$$\sum_{\mu=1}^{k} \left[\nabla^\mu u\right]_{M^{\frac{2k}{\mu}, 2k}(B_{r_0}(x_0))} \leq \epsilon,$$

we have

$$u \in \mathcal{C}^{\infty}\left(B_{\frac{r_0}{4}}(x_0), \mathcal{N}\right).$$

Remark 1.2 For $2kp > m$, Sobolev embedding implies that every map u in $W^{k,2p}(\Omega, \mathcal{N})$ is Hölder-continuous. Smoothness then follows at once from elliptic bootstrapping arguments.

In view of the monotonicity formulae for stationary harmonic and biharmonic maps, the condition on the Morrey semi-norms is natural, and in these cases, satisfied almost everywhere. More precisely, we deduce the following

Corollary 1.2 *For $k \in \{1, 2\}$ and $p > 1$, let $u \in W^{k,2p}(\Omega, \mathcal{N})$ be stationary k-harmonic. Then, u is smooth outside a closed set S with $\mathcal{H}^{m-2kp}(S) = 0$.*

Conceivably, a monotonicity formula allowing to guarantee the condition on the Morrey semi-norms in Theorem 1.1 will also hold for $k \geq 3$.

The proof of Theorem 1.1 is based on a Morrey decay estimate for the rescaled polyharmonic energy. We employ an interpolation inequality by Adams-Frazier [1] (see also Meyer-Rivière [15], Strzelecki [29] and Pumberger [20]) to bound the $W^{k,2p}$-norm by the BMO-semi-norm and the $W^{2k,p}$-norm.

The idea of proving ϵ-regularity results with interpolation inequalities first appeared in Meyer-Rivière [15] in the context of Yang-Mills fields. Recently, it has also been used by Strzelecki and Zatorska-Goldstein [30] for proving the smoothness of bounded weak solutions of a fourth order nonlinear elliptic system with critical growth under suitable smallness assumptions.

Regarding the integrability assumption, we would like to point out that the critical case $p = 1$ would be the most natural exponent for the present problem. Moreover, Corollary 1.2 directly follows from Bethuel [3] and Wang [32], respectively, applying Poincaré's inequality. Nevertheless, our proof still is of interest since it is more direct and avoids the moving frame technique.

We would like to remark that polyharmonic maps have already been studied by Gastel in [7], where he considered the polyharmonic map heat flow in the critical dimension.

2 Euler–Lagrange equation for polyharmonic maps

In this section we derive the geometric form of the Euler–Lagrange equation for weakly polyharmonic maps and analyze its structure. We consider the tubular neighborhood V_δ of $\mathcal{N}$ in $\mathbb{R}^N$, for $\delta > 0$ sufficiently small, and the smooth nearest point projection $\Pi_{\mathcal{N}} : V_\delta \longrightarrow \mathcal{N}$. For $p \in \mathcal{N}$, $P(p) := \nabla\Pi(p)$ is the orthonormal projection onto the tangent space $T_p\mathcal{N}$. The orthonormal projection onto the normal space will be denoted by $P^\perp$. Recalling $P + P^\perp = id$, we have

Lemma 2.1 (Euler–Lagrange) *If $u \in W^{k,2}(\Omega, \mathcal{N})$ is weakly k-harmonic, it satisfies*

$$P(u)(\Delta^k u) = 0 \tag{2}$$

in the sense of distributions.

Proof For $\xi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^N)$, we compute

$$0 = \frac{d}{dt}\bigg|_{t=0} \int_\Omega |\nabla^k(\Pi_{\mathcal{N}} \circ (u + t\xi))|^2 dx = 2\int_\Omega \nabla^k u \nabla^k(P(u)(\xi))dx. \qquad \square$$

Remark 2.1 For a weakly polyharmonic map u in $\mathcal{C}^\infty(\Omega,\mathcal{N})\cap W^{k,2}(\Omega,\mathcal{N})$ and $\xi\in\mathcal{C}_0^\infty(\Omega,\mathbb{R}^N)$, with $\xi(x)$ parallel to $T_{u(x)}\mathcal{N}$ for all $x\in\Omega$, we have $P(u)(\xi)=\xi$. The proof of Lemma 2.1 then shows that $\Delta^k u\perp T_u\mathcal{N}$ in the sense of distributions.

In order to formulate the following lemma, we introduce the l-divergence $\nabla^{(l)}$, by defining $\nabla^{(1)}\cdot u:=\nabla\cdot u$ and $\nabla^{(l)}\cdot u:=\nabla\cdot\left(\nabla^{(l-1)}\cdot u\right)$ for $l\geq 2$.

Lemma 2.2 *If $u\in W^{k,2}(\Omega,\mathcal{N})$ is weakly k-harmonic, there exist f and g_{jl} with*

$$\Delta^k u = f + \sum_{\substack{j,l\geq 0\\ 1\leq 2j+l\leq k}} \nabla^{(l)}\cdot\Delta^j g_{jl}, \tag{3}$$

where

$$|f|\leq C\sum_{\lambda\in\Lambda}\prod_{\mu=1}^{k}|\nabla^\mu u|^{\gamma_{\lambda,\mu}} \text{ with } \sum_\mu \mu\gamma_{\lambda,\mu}=2k \text{ for every } \lambda\in\Lambda,$$

$$|g_{jl}|\leq C\sum_{\lambda\in\Lambda_{jl}}\prod_{\mu=1}^{k}|\nabla^\mu u|^{\gamma_{\lambda,\mu}} \text{ with } \sum_\mu \mu\gamma_{\lambda,\mu}=2k-(2j+l) \text{ for every } \lambda\in\Lambda_{jl},$$

with Λ and Λ_{jl} consisting of finitely many indices and $\gamma_{\lambda,\mu}\geq 0$ for every $\lambda\in\Lambda$ (or $\lambda\in\Lambda_{jl}$) and $0\leq\mu\leq k$.

Proof We observe that

$$\Delta^k(a\cdot b)=\sum_{\substack{0\leq i,j,q\leq k\\ i+j+q=k}} c_{ijq}^k\Delta^i\nabla^q a\cdot\Delta^j\nabla^q b$$

where c_{ijq}^k are positive integers. In particular, we have $c_{k00}^k=c_{0k0}^k=1$. Combining this with Eq. 2 shows that u satisfies

$$0=P(u)(\Delta^k u)=\Delta^{k-1}(P(u)(\Delta u))-\sum_{\substack{0\leq i,j,q\leq k-1\\ i+j+q=k-1\\ (i,j,q)\neq(0,k-1,0)}} c_{ijq}^{k-1}\Delta^i\nabla^q(P(u))\Delta^{j+1}\nabla^q u$$

in the sense of distributions. Let $A(\cdot)(\cdot,\cdot)$ denote the second fundamental form of $\mathcal{N}$ in $\mathbb{R}^N$ and use the property $P(u)(\Delta u)=\Delta u+A(u)(\nabla u,\nabla u)$ to derive

$$\Delta^k u=\sum_{\substack{i,j,q\\ i+j+q=k-1\\ (i,j,q)\neq(0,k-1,0)}} c_{ijq}^{k-1}\Delta^i\nabla^q(P(u))\Delta^{j+1}\nabla^q u-\Delta^{k-1}(A(u)(\nabla u,\nabla u)). \tag{4}$$

First, we consider the case when k is even and analyze the Euler–Lagrange Eq. 4 term by term. It suffices to show that every term in (4) can be written in the desired form.

For i,q such that $i+\frac{q}{2}=\frac{k}{2}$, we have $c_{ijq}^{k-1}\Delta^i\nabla^q(P(u))\Delta^{j+1}\nabla^q u$ of the form f, where we used the fact that

$$|\nabla^\beta P(u)|\leq C\sum_{\lambda\in\Lambda}\prod_{\mu=1}^{\beta}|\nabla^\mu u|^{\gamma_{\lambda,\mu}} \quad\text{with } \sum_\mu\mu\gamma_{\lambda,\mu}=\beta \text{ for every } \beta\leq k \text{ and } \lambda\in\Lambda. \tag{5}$$

Indeed, the chain rule gives $\nabla(P(u))=\nabla P(u)\nabla u$ and $\nabla^2(P(u))=\nabla^2P(u)\nabla u\nabla u+\nabla P(u)\nabla^2 u$. We infer estimate (5) from iterating this computation, using the smoothness of

$\Pi_{\mathcal{N}}$ and observing that the L^{∞}-norm of u is bounded.

For i, q such that $i + \frac{q}{2} > \frac{k}{2}$, we compute

$$\begin{aligned}\Delta^{i}\nabla^{q}(P(u))\Delta^{j+1}\nabla^{q}u &= \nabla\cdot(\Delta^{i-1}\nabla^{q+1}(P(u))\Delta^{j+1}\nabla^{q}u)\\ &\quad-\Delta^{i-1}\nabla^{q+1}(P(u))\Delta^{j+1}\nabla^{q+1}u\end{aligned}$$

and/or

$$\Delta^{i}\nabla^{q}(P(u))\Delta^{j+1}\nabla^{q}u = \nabla\cdot(\Delta^{i}\nabla^{q-1}(P(u))\Delta^{j+1}\nabla^{q}u) - \Delta^{i}\nabla^{q-1}(P(u))\Delta^{j+2}\nabla^{q-1}u.$$

Iterating these computations from (5), we get that $c_{ijq}^{k-1}\Delta^{i}\nabla^{q}(P(u))\Delta^{j+1}\nabla^{q}u$ is of the form

$$f + \sum_{\substack{\tilde{j}, l \geq 0 \\ 1 \leq 2\tilde{j} + l \leq k}} \nabla^{(l)}\cdot\Delta^{\tilde{j}}g_{\tilde{j}l}$$

whenever $i + \frac{q}{2} > \frac{k}{2}$. The terms for i, q such that $i + \frac{q}{2} < \frac{k}{2}$ are estimated similarly. Finally, we have

$$\Delta^{k-1}(A(u)(\nabla u, \nabla u)) = \nabla\cdot\Delta^{\gamma}g_{\gamma 1}, \quad \text{with } \gamma = \frac{k}{2} - 1,$$

completing the case when k is even.

For k odd, we distinguish between the three cases $i + \frac{q}{2} = \frac{k+1}{2}$, $i + \frac{q}{2} > \frac{k+1}{2}$, and $i + \frac{q}{2} < \frac{k+1}{2}$ and proceed similarly to the case when k is even. Moreover, we get

$$\Delta^{k-1}(A(u)(\nabla u, \nabla u)) = \begin{cases} f & \text{for } k = 1\\ \Delta^{\gamma}g_{\gamma 0}, \quad \text{with } \gamma = \frac{k-1}{2} & \text{for } k \geq 3 \text{ odd.}\end{cases}$$

This, completes the proof. □

Remark 2.2 Observe that harmonic maps ($k = 1$) satisfy

$$\Delta u = -A(u)(\nabla u, \nabla u) \quad \text{in } \mathcal{D}'.$$

Thus the harmonic map equation is of the form $\Delta u = f$ with

$$|f| = |A(u)(\nabla u, \nabla u)| \leq C|\nabla u|^2.$$

Weakly biharmonic maps ($k = 2$) satisfy

$$\Delta^2 u = -\Delta P(u)\Delta u + \nabla\cdot(2\nabla P(u)\Delta u - \nabla(A(u)(\nabla u, \nabla u))) \quad \text{in } \mathcal{D}',$$

i.e., the biharmonic map equation is of the form $\Delta^2 u = f + \nabla\cdot g_{01}$ with

$$|f| = |\Delta P(u)\Delta u| \leq C(|\nabla^2 u|^2 + |\nabla^2 u||\nabla u|)$$

and

$$|g_{01}| = |2\nabla P(u)\Delta u - \nabla(A(u)(\nabla u, \nabla u))| \leq C(|\nabla^2 u||\nabla u| + |\nabla^3 u|).$$

However, we could also write the biharmonic map equation as

$$\Delta^2 u = -\Delta P(u)\Delta u + \nabla\cdot(2\nabla P(u)\Delta u) + \Delta(-A(u)(\nabla u, \nabla u)) \quad \text{in } \mathcal{D}',$$

i.e., it is also of the form $\Delta^2 u = f + \nabla \cdot g_{01} + \Delta g_{10}$ with

$$|f| = |\Delta P(u)\Delta u| \le C(|\nabla^2 u|^2 + |\nabla^2 u||\nabla u|^2), \quad |g_{01}| = |2\nabla P(u)\Delta u| \le C|\nabla^2 u||\nabla u|$$

and

$$|g_{10}| = |A(u)(\nabla u, \nabla u)| \le C|\nabla u|^2.$$

This shows that the representations of Lemma 2.2 are not unique.

3 Proof of Theorem 1.1

We will deduce Theorem 1.1 from the following

Proposition 3.1 *For $p > 1$, let $u \in W^{k,2p}(\Omega, \mathcal{N})$ be weakly k-harmonic. There exist $\epsilon > 0$ and $\tau \in (0, 1)$ such that for each point $y_0 \in \Omega$ for which there exists a radius $r_0 > 0$ with*

$$\sum_{\mu=1}^{k} \left[\nabla^\mu u\right]_{M^{\frac{2k}{\mu},2k}(B_{r_0}(y_0))} \le \epsilon,$$

we have

$$(\tau r)^{2kp-m} \sum_{\mu=1}^{k} \int_{B_{\tau r}(x_0)} |\nabla^\mu u|^{\frac{2kp}{\mu}}\, dx \le \frac{3}{4} r^{2kp-m} \sum_{\mu=1}^{k} \int_{B_r(x_0)} |\nabla^\mu u|^{\frac{2kp}{\mu}}\, dx, \tag{6}$$

for all $x_0 \in B_{r_0}(y_0)$, $0 < 4r < dist(x_0, \partial B_{r_0}(y_0))$.

Proof We let v be the k-harmonic extension of u defined as the unique solution to the following Dirichlet problem:

$$\begin{cases} \Delta^k v = 0 & \text{in } B_r(x_0) \\ \dfrac{\partial^l v}{\partial \nu^l} = \dfrac{\partial^l u}{\partial \nu^l} & \text{on } \partial B_r(x_0), \end{cases}$$

for $0 \le l \le k-1$, where ν denotes the unit normal vector to $\partial B_r(x_0)$. According to Lemma C.1 we have

$$\int_{B_\rho(x_0)} |\nabla^\mu v|^{\frac{2kp}{\mu}}\, dx \le C\left(\frac{\rho}{r}\right)^m \sum_{\lambda=1}^{k} \int_{B_{\frac{r}{4}}(x_0)} |\nabla^\lambda v|^{\frac{2kp}{\lambda}}\, dx \tag{7}$$

for $0 < \rho \le \frac{r}{4}$ and $1 \le \mu \le k$. It follows that

$$
\begin{aligned}
\sum_{\mu=1}^{k}\int_{B_\rho(x_0)} |\nabla^\mu u|^{\frac{2kp}{\mu}}dx &\le C\sum_{\mu=1}^{k}\int_{B_\rho(x_0)} |\nabla^\mu v|^{\frac{2kp}{\mu}}dx + C\sum_{\mu=1}^{k}\int_{B_\rho(x_0)} |\nabla^\mu (u-v)|^{\frac{2kp}{\mu}}dx\\
&\le C\left(\frac{\rho}{r}\right)^m\sum_{\mu=1}^{k}\int_{B_{\frac{r}{4}}(x_0)} |\nabla^\mu v|^{\frac{2kp}{\mu}}dx + C\sum_{\mu=1}^{k}\int_{B_{\frac{r}{4}}(x_0)} |\nabla^\mu (u-v)|^{\frac{2kp}{\mu}}dx\\
&\le C\left(\frac{\rho}{r}\right)^m\sum_{\mu=1}^{k}\int_{B_r(x_0)} |\nabla^\mu u|^{\frac{2kp}{\mu}}dx + C\sum_{\mu=1}^{k}\int_{B_{\frac{r}{4}}(x_0)} |\nabla^\mu (u-v)|^{\frac{2kp}{\mu}}dx.
\end{aligned}
\tag{8}
$$

In view of Lemma 2.2, we introduce the auxiliary maps u_f and $u_{g_{jl}}$ for all $j, l \ge 0$ with $1 \le 2j + l \le k$ as the solutions to the Dirichlet problems

$$
\begin{aligned}
\Delta^k u_f &= f && \text{with } u_f - u \in W_0^{k,2p}(B_r(x_0)),\\
\Delta^k u_{g_{jl}} &= \nabla^{(l)}\cdot\Delta^j g_{jl} && \text{with } u_{g_{jl}} \in W_0^{k,2p}(B_r(x_0)),
\end{aligned}
$$

where f and g_{jl} satisfy (3). Observe that the uniqueness of the Dirichlet problems implies

$$
u = u_f + \sum_{\substack{j,l\ge 0\\ 1\le 2j+l\le k}} u_{g_{jl}}. \tag{9}
$$

Moreover, $u_f - v \in W_0^{k,2p}(B_r(x_0))$ satisfies

$$
\Delta^k(u_f - v) = f.
$$

Lemma 2.2, Hölder's inequality, and Nirenberg's interpolation inequality (33) give

$$
\begin{aligned}
\|f\|_{L^p(B_r(x_0))} &\le C\sum_{\lambda\in\Lambda}\prod_{\mu=1}^{k}\left\|\nabla^\mu u\right\|^{\gamma_{\lambda,\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))}\\
&\le C\sum_{\lambda\in\Lambda}\prod_{\mu=1}^{k}\|u\|^{\gamma_{\lambda,\mu}(1-\frac{\mu}{k})}_{L^\infty(B_r(x_0))}\|u\|^{\frac{\mu\gamma_{\lambda,\mu}}{k}}_{W^{k,2p}(B_r(x_0))}\\
&\le C\|u\|^2_{W^{k,2p}(B_r(x_0))} < \infty,
\end{aligned}
$$

and similarly, we obtain

$$
\|g_{jl}\|_{L^{r_{jl}}(B_r(x_0))} \le C\|u\|^{\eta_{jl}}_{W^{k,2p}(B_r(x_0))} < \infty, \tag{10}
$$

where

$$
r_{jl} = \frac{2kp}{2k-(2j+l)} \quad\text{and}\quad 1 \le \eta_{jl} := \frac{2k-(2j+l)}{k} \le 2.
$$

Thus, Lemma B.1 and Lemma B.3 imply that

$$
u_f - v \in W^{2k,p}(B_r(x_0)),\quad u_{g_{jl}} \in W^{2k-(2j+l),r_{jl}}(B_r(x_0)),
$$

and

$$
\begin{aligned}
\left\|\nabla^{2k}(u_f - v)\right\|_{L^p(B_r(x_0))} &\le C\|f\|_{L^p(B_r(x_0))},\\
\left\|\nabla^{2k-(2j+l)}u_{g_{jl}}\right\|_{L^{r_{jl}}(B_r(x_0))} &\le C\|g_{jl}\|_{L^{r_{jl}}(B_r(x_0))}.
\end{aligned}
\tag{11}
$$

We remark that the only place where we need $p > 1$ is to ensure the first estimate for $u_f - v$. From (9), we get

$$\int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde{\mu}}(u-v)|^{\frac{2kp}{\tilde{\mu}}}\,dx \le C \int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde{\mu}}(u_f-v)|^{\frac{2kp}{\tilde{\mu}}}\,dx$$

$$+C \sum_{\substack{j,l\ge 0\\ 1\le 2j+l\le k}} \int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde{\mu}}u_{g_{jl}}|^{\frac{2kp}{\tilde{\mu}}}\,dx \tag{12}$$

for $1 \le \tilde{\mu} \le k$. We apply the Gagliardo–Nirenberg interpolation inequality (34), Lemma B.2, estimates (11), and Lemma 2.2 to obtain

$$\begin{aligned}
\left\|\nabla^{\tilde{\mu}}(u_f-v)\right\|_{L^{\frac{2kp}{\tilde{\mu}}}(B_{\frac{r}{4}}(x_0))} &\le C\left[u_f-v\right]^{1-\frac{\tilde{\mu}}{2k}}_{BMO(B_{\frac{r}{4}}(x_0))}\|u_f-v\|^{\frac{\tilde{\mu}}{2k}}_{W^{2k,p}(B_{\frac{r}{4}}(x_0))}\\
&\le C\left[u_f-v\right]^{1-\frac{\tilde{\mu}}{2k}}_{BMO(B_{\frac{r}{4}}(x_0))}\|\nabla^{2k}(u_f-v)\|^{\frac{\tilde{\mu}}{2k}}_{L^p(B_r(x_0))}\\
&\le C\left[u_f-v\right]^{1-\frac{\tilde{\mu}}{2k}}_{BMO(B_{\frac{r}{4}}(x_0))}\|f\|^{\frac{\tilde{\mu}}{2k}}_{L^p(B_r(x_0))}\\
&\le C\left[u_f-v\right]^{1-\frac{\tilde{\mu}}{2k}}_{BMO(B_{\frac{r}{4}}(x_0))}\left\|\sum_{\lambda\in\Lambda}\prod_{\mu=1}^{k}|\nabla^{\mu}u|^{\gamma_{\lambda,\mu}}\right\|^{\frac{\tilde{\mu}}{2k}}_{L^p(B_r(x_0))},
\end{aligned}$$

with $\sum_{\mu}\mu\gamma_{\lambda,\mu} = 2k$ for every $\lambda \in \Lambda$, where a resacling argument shows that the constant C is independent of r. Next, we use Hölder's and Young's inequalities to derive

$$\left\|\sum_{\lambda\in\Lambda}\prod_{\mu=1}^{k}|\nabla^{\mu}u|^{\gamma_{\lambda,\mu}}\right\|_{L^p(B_r(x_0))} \le \sum_{\lambda\in\Lambda}\prod_{\mu=1}^{k}\left\|\nabla^{\mu}u\right\|^{\gamma_{\lambda,\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))} \le C\sum_{\mu=1}^{k}\left\|\nabla^{\mu}u\right\|^{\frac{2k}{\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))},$$

where we remark that $\sum_{\mu=1}^{k}\frac{\mu\gamma_{\lambda,\mu}}{2k} = 1$. Combining the above estimates, we obtain

$$\int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde{\mu}}(u_f-v)|^{\frac{2kp}{\tilde{\mu}}}\,dx \le C\left[u_f-v\right]^{\left(1-\frac{\tilde{\mu}}{2k}\right)\frac{2kp}{\tilde{\mu}}}_{BMO(B_{\frac{r}{4}}(x_0))}\sum_{\mu=1}^{k}\int_{B_r(x_0)}|\nabla^{\mu}u|^{\frac{2kp}{\mu}}\,dx. \tag{13}$$

For the second term in (12), as before, the Gagliardo–Nirenberg interpolation inequality (34) gives

$$\left\|\nabla^{\tilde{\mu}}u_{g_{jl}}\right\|_{L^{\frac{2kp}{\tilde{\mu}}}(B_{\frac{r}{4}}(x_0))} \le C\left[u_{g_{jl}}\right]^{1-\frac{\tilde{\mu}}{k}\theta_{jl}}_{BMO(B_{\frac{r}{4}}(x_0))}\left\|u_{g_{jl}}\right\|^{\frac{\tilde{\mu}}{k}\theta_{jl}}_{W^{2k-(2j+l),r_{jl}}(B_{\frac{r}{4}}(x_0))}, \tag{14}$$

where

$$\frac{1}{2} \le \theta_{jl} := \frac{k}{2k-(2j+l)} = \eta_{jl}^{-1} \le 1$$

for all $j,l \ge 0$ with $1 \le 2j+l \le k$. Furthermore, we again apply Lemma B.2 with $\mu = 2k-(2j+l)$, estimates (11), Lemma 2.2, Hölder's inequality, and Young's inequality to get

$$\begin{aligned}\left\|u_{g_{jl}}\right\|_{W^{2k-(2j+l),r_{jl}}(B_r(x_0))} &\leq \left\|\nabla^{2k-(2j+l)}u_{g_{jl}}\right\|_{L^{r_{jl}}(B_r(x_0))}\\ &\leq C\left\|g_{jl}\right\|_{L^{r_{jl}}(B_r(x_0))}\\ &\leq C\sum_{\lambda\in\Lambda}\prod_{\mu=1}^{k}\left\|\nabla^{\mu}u\right\|^{\gamma_{\lambda,\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))}\\ &\leq C\sum_{\mu=1}^{k}\left\|\nabla^{\mu}u\right\|^{\frac{k\eta_{jl}}{\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))},\end{aligned}\tag{15}$$

where

$$\eta_{jl} := \frac{2k-(2j+l)}{k} = \theta_{jl}^{-1}.$$

Combining (14) and (15) gives

$$\int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde{\mu}}u_{g_{jl}}|^{\frac{2kp}{\tilde{\mu}}}\,dx \leq C\left[u_{g_{jl}}\right]_{BMO(B_{\frac{r}{4}}(x_0))}^{\left(1-\frac{\tilde{\mu}}{k}\theta_{jl}\right)\frac{2kp}{\tilde{\mu}}}\sum_{\mu=1}^{k}\int_{B_r(x_0)}|\nabla^{\mu}u|^{\frac{2kp}{\mu}}\,dx.\tag{16}$$

From Lemma 3.3 below, we infer

$$\left[u_f-v\right]_{BMO(B_{\frac{r}{4}}(x_0))}^{\left(1-\frac{\tilde{\mu}}{2k}\right)\frac{2kp}{\tilde{\mu}}}+\sum_{j,l}\left[u_{g_{jl}}\right]_{BMO(B_{\frac{r}{4}}(x_0))}^{\left(1-\frac{\tilde{\mu}}{k}\theta_{jl}\right)\frac{2kp}{\tilde{\mu}}}\leq C\epsilon^{\beta}\tag{17}$$

for some $\beta>0$, all $1\leq\tilde{\mu}\leq k$ and all $j,l\geq 0$ with $1\leq 2j+l\leq k$. Thus, from (12)–(13) and (16)–(17) we conclude

$$\int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde{\mu}}(u-v)|^{\frac{2kp}{\tilde{\mu}}}\,dx \leq C\epsilon^{\beta}\sum_{\mu=1}^{k}\int_{B_r(x_0)}|\nabla^{\mu}u|^{\frac{2kp}{\mu}}\,dx.\tag{18}$$

Finally, we combine inequalities (8) and (18) into

$$\begin{aligned}&\rho^{2kp-m}\sum_{\mu=1}^{k}\int_{B_\rho(x_0)}|\nabla^{\mu}u|^{\frac{2kp}{\mu}}\,dx\\ &\quad\leq C_1\left(\left(\frac{\rho}{r}\right)^{2kp}+\epsilon^{\beta}\left(\frac{\rho}{r}\right)^{2kp-m}\right)r^{2kp-m}\sum_{\mu=1}^{k}\int_{B_r(x_0)}|\nabla^{\mu}u|^{\frac{2kp}{\mu}}\,dx.\end{aligned}$$

We conclude the proof of this proposition by setting $\tau:=\frac{\rho}{\tau}$ equal to $(2C_1)^{-\frac{1}{2kp}}$ and choosing $\epsilon>0$ sufficiently small so that $C_1\epsilon^{\beta}\tau^{2kp-m}\leq\frac{1}{4}$. □

It remains to show (17) for which, as a first step, we prove the subsequent lemma.

Lemma 3.2 *There exists a constant $C > 1$ such that*

$$\sum_{\mu=1}^{k} [\nabla^{\mu} u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_r(x_0))} \leq C \sum_{\mu=1}^{k} [\nabla^{\mu} u]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} + Cr^{2k-m} \sum_{\mu=1}^{k} \int_{B_r(x_0)} |\nabla^{\mu} u_{g_{jl}}|^{\frac{2k}{\mu}} dx. \tag{19}$$

Proof For $B_s(x) \subset B_{4r}(x_0)$, we consider the k-harmonic extension v_{jl} of $u_{g_{jl}}$ on $B_s(x)$, where $u_{g_{jl}}$ is set to zero outside $B_r(x_0)$. Then, $w_{jl} := u_{g_{jl}} - v_{jl} \in W_0^{k,2}(B_s(x))$ satisfies

$$\Delta^k w_{jl} = \nabla^{(l)} \cdot \Delta^j \tilde{g}_{jl},$$

where $\tilde{g}_{jl} := \chi_{B_r(x_0)} g_{jl}$. Analogous to (10), we conclude that $w_{jl} \in W^{2k-(2j+l),r_{jl}}(B_s(x))$, and similar to (15), we get

$$\|w_{jl}\|_{W^{2k-(2j+l),r_{jl}}(B_s(x))} \leq C \sum_{\mu=1}^{k} \|\nabla^{\mu} u\|^{\frac{k\eta_{jl}}{\mu}}_{L^{\frac{2k}{\mu}}(B_s(x))}.$$

Here and henceforth in the proof of Lemma 3.2, we set $p = 1$ and observe that the second estimate in (11) is still valid in this case.

Applying the Gagliardo–Nirenberg inequality (34) and the preceding estimate gives

$$\begin{aligned} \int_{B_{\frac{s}{4}}(x)} |\nabla^{\lambda} w_{jl}|^{\frac{2k}{\lambda}} dx &\leq C [w_{jl}]^{2k\left(\frac{1}{\lambda} - \frac{1}{2k-(2j+l)}\right)}_{BMO(B_{\frac{s}{4}}(x))} \sum_{\mu=0}^{2k-(2j+l)} \int_{B_{\frac{s}{4}}(x)} |\nabla^{\mu} w_{jl}|^{r_{jl}} dx \\ &\leq C [w_{jl}]^{2k\left(\frac{1}{\lambda} - \frac{1}{2k-(2j+l)}\right)}_{BMO(B_{\frac{s}{4}}(x))} \sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^{\mu} u|^{\frac{2k}{\mu}} dx \end{aligned} \tag{20}$$

for $1 \leq \lambda \leq k$. As v_{jl} is the k-harmonic extension of $u_{g_{jl}}$, we obtain with Hölder's inequality, Poincaré's inequality, and Lemma C.2

$$\begin{aligned} [w_{jl}]^{2k}_{BMO(B_{\frac{s}{4}}(x))} &\leq \sup_{B \subset B_{\frac{s}{2}}(x)} \left\{ \fint_B |w_{jl} - \overline{w_{jl}}_B| dx \right\}^{2k} \\ &\leq C [\nabla w_{jl}]^{2k}_{M^{2,2}(B_{\frac{s}{2}}(x))} \\ &\leq C [\nabla u_{g_{jl}}]^{2k}_{M^{2,2}(B_{\frac{s}{2}}(x))} + C [\nabla v_{jl}]^{2k}_{M^{2,2}(B_{\frac{s}{2}}(x))} \\ &\leq C [\nabla u_{g_{jl}}]^{2k}_{M^{2,2}(B_{\frac{s}{2}}(x))} + C \left(\sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^{\mu} u_{g_{jl}}|^{\frac{2k}{\mu}} dx \right)^{\mu}. \end{aligned} \tag{21}$$

Arguing with the help of Poincaré's inequality and (10), we show that

$$\begin{aligned} \|u_{g_{jl}}\|_{W^{k,2}(B_r(x_0))} &\leq C \|\nabla^k u_{g_{jl}}\|_{L^2(B_r(x_0))} \leq C \|g_{jl}\|_{L^2(B_r(x_0))} \\ &\leq C \|u\|_{W^{k,2}(B_r(x_0))} \leq C\epsilon \leq 1, \end{aligned}$$

provided $\epsilon > 0$ is sufficiently small. Thus, we can omit the exponent μ in (21) and estimate

$$[w_{jl}]^{2k}_{BMO(B_{\frac{s}{4}}(x))} \leq C \sum_{\mu=1}^{k} [\nabla^{\mu} u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))}. \tag{22}$$

Combining estimates (20) and (22) with Young's inequality yields

$$\begin{aligned}\int_{B_{\frac{s}{4}}(x)} |\nabla^{\lambda} w_{jl}|^{\frac{2k}{\lambda}} dx &\leq C \left(\sum_{\mu=1}^{k} [\nabla^{\mu} u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))} \right)^{\left(\frac{1}{\lambda} - \frac{1}{2k-(2j+l)}\right)} \sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^{\mu} u|^{\frac{2k}{\mu}} dx \\ &\leq C\gamma \sum_{\mu=1}^{k} [\nabla^{\mu} u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))} + C(\gamma) \sum_{\mu=1}^{k} \|\nabla^{\mu} u\|^{\frac{2k\overline{C}(\lambda)}{\mu}}_{L^{\frac{2k}{\mu}}(B_s(x))},\end{aligned} \tag{23}$$

with $\gamma > 0$ and $\overline{C}(\lambda) > 1$ for $1 \leq 2j + l,\ \lambda \leq k$. With a rescaling argument, this gives

$$\begin{aligned}\int_{B_{\frac{s}{4}}(x)} |\nabla^{\lambda} w_{jl}|^{\frac{2k}{\lambda}} dx &\leq C s^{m-2k} \gamma \sum_{\mu=1}^{k} [\nabla^{\mu} u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))} \\ &\quad + C(\gamma) \sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^{\mu} u|^{\frac{2k}{\mu}} dx,\end{aligned} \tag{24}$$

where the constants are now independent of s and λ. Here, we also used that $\|\nabla^{\mu} u\|_{L^{\frac{2k}{\mu}}(B_{4r}(x_0))} \leq 1$ for $1 \leq \mu \leq k$, provided $\epsilon > 0$ is sufficiently small. Combining estimate (24) with Lemma C.1, we estimate for $0 < \rho \leq \frac{s}{4}$ as in (8)

$$\begin{aligned}\sum_{\mu=1}^{k} \int_{B_{\rho}(x)} |\nabla^{\mu} u_{g_{jl}}|^{\frac{2k}{\mu}} dx &\leq C \left(\frac{\rho}{s}\right)^{m} \sum_{\mu=1}^{k} \int_{B_{\frac{s}{4}}(x)} |\nabla^{\mu} u_{g_{jl}}|^{\frac{2k}{\mu}} dx + C \sum_{\mu=1}^{k} \int_{B_{\frac{s}{4}}(x)} |\nabla^{\mu} w_{jl}|^{\frac{2k}{\mu}} dx \\ &\leq C \left(\frac{\rho}{s}\right)^{m} \sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^{\mu} u_{g_{jl}}|^{\frac{2k}{\mu}} dx + C(\gamma) \sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^{\mu} u|^{\frac{2k}{\mu}} dx \\ &\quad + C s^{m-2k} \gamma \sum_{\mu=1}^{k} [\nabla^{\mu} u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))}.\end{aligned} \tag{25}$$

The proof of the lemma is completed by the following iteration argument. To simplify notation, we define $T(\rho) := \rho^{2k-m} \sum_{\mu=1}^{k} \int_{B_{\rho}(x)} |\nabla^{\mu} u_{g_{jl}}|^{\frac{2k}{\mu}} dx$ so that the above estimate becomes

$$\begin{aligned}T(\rho) &\leq C \left(\frac{\rho}{s}\right)^{2k} T(s) + C(\gamma) \left(\frac{\rho}{s}\right)^{2k-m} s^{2k-m} \sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^{\mu} u|^{\frac{2k}{\mu}} dx \\ &\quad + C\gamma \left(\frac{\rho}{s}\right)^{2k-m} \sum_{\mu=1}^{k} [\nabla^{\mu} u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))}.\end{aligned} \tag{26}$$

From now on, we choose $\tau := \frac{\rho}{s}$ sufficiently small such that $C\tau^{2k} \leq \frac{1}{2}$. For any $B_\sigma(x) \subset B_{2r}(x_0)$, there exists $i \in \mathbb{N}$ (simply set $i = \lfloor \log \frac{\sigma}{2r} / \log \tau \rfloor$) with $\tau^{i+1}2r \leq \sigma \leq \tau^i 2r$, and therefore, $(\tau^i 2r)^{2k-m} \leq \sigma^{2k-m} \leq (\tau^{(i+1)}2r)^{(2k-m)}$. Estimate (26) then gives

$$T(\tau^i 2r) \leq \frac{1}{2} T(\tau^{i-1}2r) + S,$$

with

$$S := C(\gamma)\tau^{2k-m} \sum_{\mu=1}^{k} [\nabla^\mu u]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} + C\gamma\tau^{2k-m} \sum_{\mu=1}^{k} [\nabla^\mu u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_r(x_0))}.$$

Iterating this inequality gives

$$T(\tau^i 2r) \leq T(2r) + \sum_{\tilde{\mu}=1}^{i} \frac{1}{2^{\tilde{\mu}}} S \leq T(2r) + S,$$

from which we have

$$\begin{aligned} T(\sigma) &\leq CT(\tau^i 2r) \\ &\leq C(\gamma) \sum_{\mu=1}^{k} [\nabla^\mu u]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} + Cr^{2k-m} \sum_{\mu=1}^{k} \int_{B_r(x_0)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}} dx \\ &\quad + C\gamma \sum_{\mu=1}^{k} [\nabla^\mu u_{g_{jl}}]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_r(x_0))}. \end{aligned}$$

The desired result now follows from taking the supremum over all such balls $B_\sigma(x)$, and choosing $\gamma > 0$ sufficiently small to absorb the last term on the right-hand side. □

Now we are able to complete the proof of Proposition 3.1 with the following

Lemma 3.3 *Assume that*

$$\sum_{\mu=1}^{k} [\nabla^\mu u]_{M^{\frac{2k}{\mu},2k}(B_r(x_0))} \leq \epsilon.$$

Then, we have

$$[u_f - v]_{BMO(B_{\frac{r}{4}}(x_0))} + \sum_{j,l} [u_{g_{jl}}]_{BMO(B_{\frac{r}{4}}(x_0))} \leq C\epsilon^\beta$$

for some $\beta > 0$ and all $j, l \geq 0$ such that $1 \leq 2j + l \leq k$.

Proof Similar to (20) and (23), we estimate

$$\int_{B_r(x_0)} |\nabla^\lambda u_{g_{jl}}|^{\frac{2k}{\lambda}} dx \leq C\gamma [u_{g_{jl}}]^{2k}_{BMO(B_r(x_0))} + C(\gamma) \sum_{\mu=1}^{k} \|\nabla^\mu u\|^{\frac{2k}{\mu}}_{L^{\frac{2k}{\mu}}(B_r(x_0))} \tag{27}$$

for $1 \leq \lambda \leq k$ and every $\gamma > 0$. Moreover, Poincaré's inequality implies

$$[u_{g_{jl}}]^{2k}_{BMO(B_r(x_0))} \leq C [u_{g_{jl}}]^{2k}_{BMO(\mathbb{R}^m)} \leq C [\nabla u_{g_{jl}}]^{2k}_{M^{2k,2k}(B_r(x_0))}.$$

Combining this with Lemma 3.2 and estimate (27) gives

$$\left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2k,2k}(B_r(x_0))} \le C\gamma \left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2k,2k}(B_r(x_0))} + C(\gamma) \sum_{\mu=1}^{k} \left[\nabla^{\mu} u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))}$$

which for $\gamma > 0$ sufficiently small implies

$$\left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2k,2k}(B_r(x_0))} \le C \sum_{\mu=1}^{k} \left[\nabla^{\mu} u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))}.$$

Applying Hölder's inequality and Poincaré's inequality together with the above estimate, we infer

$$\left[u_{g_{jl}}\right]^{2k}_{BMO(B_{\frac{r}{4}}(x_0))} \le C \left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2k,2k}(B_r(x_0))} \le C \sum_{\mu=1}^{k} \left[\nabla^{\mu} u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} \le C\epsilon^{\beta} \tag{28}$$

for some $\beta > 0$. From (9), we deduce

$$\left[u_f - v\right]_{BMO(B_{\frac{r}{4}}(x_0))} \le C \sum_{j,l\ge 0} \left[u_{g_{jl}}\right]_{BMO(B_{\frac{r}{4}}(x_0))} + C\left[u - v\right]_{BMO(B_{\frac{r}{4}}(x_0))}. \tag{29}$$

Hölder's inequality and Poincaré's inequality imply

$$\begin{aligned}[u - v]^{2k}_{BMO(B_{\frac{r}{4}}(x_0))} &\le C\left[\nabla(u - v)\right]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))} \\ &\le C\left([\nabla u]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))} + [\nabla v]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))}\right).\end{aligned} \tag{30}$$

As v is the k-harmonic extension of u, Lemma C.2 and Hölder's inequality imply

$$\begin{aligned}[\nabla u]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))} + [\nabla v]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))} &\le C \sum_{\mu=1}^{k} \left[\nabla^{\mu} u\right]^{2k}_{M^{2,2\mu}(B_r(x_0))} \\ &\le C \sum_{\mu=1}^{k} \left[\nabla^{\mu} u\right]^{2k}_{M^{\frac{2k}{\mu},2k}(B_r(x_0))} \\ &\le C\epsilon^{\beta}\end{aligned} \tag{31}$$

for some $\beta > 0$. Estimate (17) now follows from (28)–(31), which completes the proof. □

To finish the proof of Theorem 1.1, note that Proposition 3.1 implies

$$\rho^{2kp-m} \sum_{\mu=1}^{k} \int_{B_{\rho}(x_0)} |\nabla^{\mu} u|^{\frac{2kp}{\mu}}\, dx \le C\rho^{\gamma}$$

for $\gamma > 0$ and all $B_{\rho}(x_0) \subset B_{\frac{r_0}{4}}(y_0)$. Hence, by Morrey's Dirichlet growth theorem in [17, Theorem 3.5.2], we conclude that $u \in C^{0,\frac{\gamma}{p}}$ in a neighborhood of y_0. The smoothness of u near y_0 then follows from elliptic bootstrapping arguments, which completes the proof of Theorem 1.1.

4 The harmonic and biharmonic cases

Here we give a derivation of Corollary 1.2. For stationary harmonic maps, i.e., $k = 1$, Corollary 1.2 follows from Theorem 1.1 and

Proposition 4.1 (Monotonicity formula [19]) *For $u \in W^{1,2}(B_2, \mathcal{N})$ stationary harmonic and $0 \le \rho \le r \le 1$, we have*

$$\rho^{2-m} \int\limits_{B_\rho} |\nabla u|^2 dx \le r^{2-m} \int\limits_{B_r} |\nabla u|^2 dx. \tag{32}$$

Indeed, consider $u \in W^{1,2p}(\Omega, \mathbb{R}^N)$ stationary harmonic and define the set

$$S := \left\{ x_0 \in \Omega : \limsup_{r \to 0} r^{2p-m} \int\limits_{B_r(x_0)} |\nabla u|^{2p} dx \ge \gamma^p \right\},$$

with $\gamma > 0$ small. We have by Ziemer [33, Corollary 3.2.3.] that $\mathcal{H}^{m-2p}(S) = 0$. Applying Hölder's inequality, we get that for any $y_0 \in \Omega \setminus S$, there exists $R > 0$ s.t.

$$R^{2-m} \int\limits_{B_R(y_0)} |\nabla u|^2 dx \le C \left(R^{2p-m} \int\limits_{B_R(y_0)} |\nabla u|^{2p} dx \right)^{\frac{1}{p}} < \gamma.$$

Hence, the monotonicity formula (32) for $x_0 \in B_{\frac{R}{2}}(y_0)$ implies

$$\begin{aligned} \rho^{2-m} \int\limits_{B_\rho(x_0)} |\nabla u|^2 dx &\le C R^{2-m} \int\limits_{B_{\frac{R}{2}}(x_0)} |\nabla u|^2 dx \\ &\le C \int\limits_{B_R(y_0)} |\nabla u|^2 dx \\ &\le C_2 \gamma. \end{aligned}$$

Fixing $\gamma := \frac{\epsilon}{C_2}$, where ϵ is given by Theorem 1.1, the claim follows.

For stationary biharmonic maps, i.e., $k = 2$, we replace Proposition 4.1 by

Proposition 4.2 (Monotonicity formula [5,2]) *For $u \in W^{2,2}(B_2, \mathcal{N})$ (extrinsically) stationary biharmonic and a.e. $0 < \rho < r \le 1$, we have*

$$r^{4-m} \int\limits_{B_r} |\Delta u|^2 dx - \rho^{4-m} \int\limits_{B_\rho} |\Delta u|^2 dx = P + R,$$

where

$$P = 4 \int\limits_{B_r \setminus B_\rho} \left(\frac{(u_j + x^i u_{ij})^2}{|x|^{m-2}} + \frac{(m-2)(x^i u_i)^2}{|x|^m} \right) dx$$

$$R = 2 \int\limits_{\partial B_r \setminus \partial B_\rho} \left(-\frac{x^i u_j u_{ij}}{|x|^{m-3}} + 2\frac{(x^i u_i)^2}{|x|^{m-1}} - 2\frac{|\nabla u|^2}{|x|^{m-3}} \right) d\sigma.$$

We observe that Nirenberg's interpolation inequality (33) implies that $\nabla u \in L^{4p}(\Omega)$ and define

$$S := \left\{ x_0 \in \Omega : \limsup_{r \to 0} r^{4p-m} \int\limits_{B_r(x_0)} \left(|\nabla^2 u|^{2p} + |\nabla u|^{4p} \right) dx \geq \eta^p \right\}$$

with $\eta > 0$ small. We have by Ziemer [33, Corollary 3.2.3.] that $\mathcal{H}^{m-4p}(S) = 0$. For any $y_0 \in \Omega \setminus S$ there exists $R > 0$ s.t.

$$R^{4-m} \int_{B_R(y_0)} (|\nabla u|^4 + |\nabla^2 u|)^2 dx \leq C \left(R^{4p-m} \int\limits_{B_R(y_0)} (|\nabla u|^{4p} + |\nabla^2 u|^{2p}) dx \right)^{\frac{1}{p}} < \eta.$$

As in the paper of Chang, et al. [5], the monotonicity formula implies [5, Corollary 4.4]. Combining this with [32, Lemma 5.4] (replacing [5, Corollary 4.7]), we can pursue the iteration argument in [5, Lemma 4.8] to conclude the existence of $\rho_0 > 0$ and $\overline{\epsilon}_0 > 0$ such that if $R^{4-m} \int_{B_R(y_0)} (|\nabla u|^4 + |\nabla^2 u|^2) dx \leq \overline{\epsilon} \leq \overline{\epsilon}_0$, we have

$$\rho^{4-m} \int\limits_{B_\rho(x_0)} (|\nabla u|^4 + |\nabla^2 u|^2) dx \leq C_3 \overline{\epsilon}$$

for all $B_\rho(x_0) \subset B_{\rho_0}(y_0)$. See also Struwe [27]. Fix $\overline{\epsilon} := \min(\frac{\epsilon}{C_3}, \overline{\epsilon}_0) > 0$ and $\eta := \overline{\epsilon} > 0$, where $\epsilon > 0$ is given by Theorem 1.1. This completes the proof.

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Appendixes

Appendix A: Gagliardo–Nirenberg inequalities

The following interpolation inequality was proven by Nirenberg in [18].

Theorem A.1 (Gagliardo–Nirenberg type inequality 1) *For $k \in \mathbb{N}$ and $1 < q, r \leq \infty$, let $u \in W^{k,r}(\mathbb{R}^m) \cap L^q(\mathbb{R}^m)$. Then, for $0 \leq j < k$, there exists $C > 0$ independent of u such that*

$$\|\nabla^j u\|_{L^p(\mathbb{R}^m)} \leq C \|\nabla^k u\|^a_{L^r(\mathbb{R}^m)} \|u\|^{1-a}_{L^q(\mathbb{R}^m)}, \tag{33}$$

where

$$\frac{1}{p} - \frac{j}{m} = a \left(\frac{1}{r} - \frac{k}{m} \right) + (1-a) \frac{1}{q},$$

for all

$$\frac{j}{k} \leq a \leq 1.$$

Remark A.1 For a bounded domain Ω with smooth boundary, the result remains true if we add the term $C\|u\|_{L^{\tilde{q}}(\Omega)}$ for any $\tilde{q} > 0$ to the right side of (33). The constants then also depend on Ω.

In particular, we infer from Theorem A.1 that for $a = \frac{1}{2}$ and $q = \infty$

$$\|\nabla^j u\|^2_{L^p(\mathbb{R}^m)} \leq C\|\nabla^k u\|_{L^r(\mathbb{R}^m)}\|u\|_{L^\infty(\mathbb{R}^m)},$$

where

$$\frac{1}{p} - \frac{j}{m} = \frac{1}{2r} - \frac{k}{2m}.$$

However, for our purposes in Sect. 3, this is not sharp enough, whence we need to employ an improved version, where the L^∞-norm is substituted by the BMO-seminorm. Such an inequality first appeared in Adams-Frazier [1] and also in Meyer-Rivière [15], Strzelecki [29], and Pumberger [20], where the following version of the Gagliardo–Nirenberg type inequality is stated.

Theorem A.2 (Gagliardo–Nirenberg type inequality 2) *Assume that $u \in W^{k,r}(\Omega)$ for some $r > 1$ and $1 \leq j < k$, with $j, k \in \mathbb{N}$. If $u \in BMO(\Omega)$, then $\nabla^j u \in L^p(\Omega)$ for $p := \frac{k}{j} r$ and*

$$\|\nabla^j u\|_{L^p(\Omega)} \leq C\,[u]^{1-\theta}_{BMO(\Omega)}\,\|u\|^\theta_{W^{k,r}(\Omega)}, \tag{34}$$

where $\theta := \frac{j}{k}$, for some constant $C = C(k, j, r, \Omega)$.

Remark A.2 When $\Omega = B_s(x_0)$ for some radius $s > 0$, Eq. 34 becomes

$$\left(s^{jp-m}\int_{B_s(x_0)}|\nabla^j u|^p dx\right)^{\frac{1}{p}} \leq C\,[u]^{1-\theta}_{BMO(B_s(x_0))}\left(\sum_{\mu=0}^k s^{\mu r-m}\int_{B_s(x_0)}|\nabla^\mu u|^r dx\right)^{\frac{\theta}{r}},$$

where the constant C is independent of s.

Appendix B: Linear estimates

For $m \geq 2k+1$, the fundamental solution of Δ^k on $\mathbb{R}^m$ is

$$\Gamma_k(x-y) = c|x-y|^{2k-m},$$

i.e.,

$$\Delta^k\Gamma_k(x-y) = \delta(x-y) \quad \text{for } x, y \in \mathbb{R}^m.$$

The kernel $K := \nabla^{2k}\Gamma_k$ verifies the hypotheses of Stein [26, Theorem II.3.2]:

Lemma B.1 *Let $f \in L^p(\Omega)$, $1 < p < \infty$, $K := \nabla^{2k}\Gamma_k$, and $u := K \star f$, which is the convolution of f by K. Then, $u \in W^{2k,p}(\Omega)$*

$$\Delta^k u = f \text{ a.e.}$$

and there exists $C > 0$ depending only on n and p such that

$$\|\nabla^{2k}u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Furthermore, we have

Lemma B.2 *For* $1 < p < \infty$, $\mu \in \mathbb{N} \cap (k, 2k]$, *and* $u \in W^{\mu,p}(\Omega) \cap W_0^{k,p}(\Omega)$ *there exists a constant* C *(independent of* u*) such that*

$$\|u\|_{W^{\mu,p}(\Omega)} \leq C\|\Delta^{\frac{\mu}{2}} u\|_{L^p(\Omega)} \text{ for } \mu \text{ even,}$$

and

$$\|u\|_{W^{\mu,p}(\Omega)} \leq C\|\nabla\Delta^{\frac{\mu-1}{2}} u\|_{L^p(\Omega)} \text{ for } \mu \text{ odd.}$$

Proof The proof is completely analogous to Gilbarg–Trudinger [10, Lemma 9.17]. □

Lemma B.3 *For a ball* $B \subset \mathbb{R}^m$, $g \in L^r(B)$ *with* $1 < r < \infty$, $k \geq 1$, *and* $j, l \geq 0$ *with* $1 \leq l + 2j \leq k$, *there exists a unique weak solution* $u \in W_0^{k,2}(B, \mathbb{R}^N)$ *of*

$$\Delta^k u = \nabla^{(l)} \cdot \Delta^j g$$

satisfying

$$\|\nabla^{2k-(2j+l)} u\|_{L^r(B)} \leq C\|g\|_{L^r(B)}.$$

Proof The proof is similar to Gilbarg–Trudinger [10] and Giaquinta [8,9]. In the case $r = 2$, existence of a unique solution follows from the Lax-Milgram Theorem, and using the method of difference quotients, we also infer the existence of higher derivatives. Following the arguments of Giaquinta [8], we conclude Lemma B.3 in this case. Stampacchia's interpolation theorem (see [9, Theorem 4.6]) then states that the claim remains true for $2 < r < \infty$ and a duality argument completes the proof. □

Appendix C: Decay Lemma

Lemma C.1 *Let* $u \in W^{k,2}(\Omega)$ *be a weakly* k*-harmonic function with* $\|u\|_{W^{k,p}(\Omega)} \leq 1$. *For* $x_0 \in \Omega$, $0 < \rho \leq r \leq dist(x_0, \partial\Omega)$, *and* $2 \leq p < \infty$, *we have* $C > 0$ *independent of* u *and* ρ *such that*

$$\rho^{-m} \sum_{l=1}^{k} \int_{B_\rho(x_0)} |\nabla^l u|^{\frac{kp}{l}} dx \leq C r^{-m} \sum_{l=1}^{k} \int_{B_r(x_0)} |\nabla^l u|^{\frac{kp}{l}} dx.$$

Proof The proof is similar to Giaquinta [8]. □

Lemma C.2 *For* $r > 0$, *we consider* $u \in W^{k,2}(B_r)$ *and* v *solving the Dirichlet problem*

$$\begin{cases} \Delta^k v = 0 \\ u - v \in W_0^{k,2}(B_r). \end{cases}$$

Then, there exists $C > 0$ *independent of* u *and* r *with*

$$\sum_{\mu=1}^{k} [\nabla^\mu v]^2_{M^{2,2\mu}(B_{\frac{r}{2}})} \leq C \sum_{\mu=1}^{k} \int_{B_r} |\nabla^\mu u|^2 dx.$$

Proof The proof is again similar to Giaquinta [8]. □

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